

# STRONG EDGE-COLORINGS OF SPARSE GRAPHS WITH LARGE MAXIMUM DEGREE

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**ABSTRACT.** A *strong  $k$ -edge-coloring* of a graph  $G$  is a mapping from  $E(G)$  to  $\{1, 2, \dots, k\}$  such that every two adjacent edges or two edges adjacent to the same edge receive distinct colors. The *strong chromatic index*  $\chi'_s(G)$  of a graph  $G$  is the smallest integer  $k$  such that  $G$  admits a strong  $k$ -edge-coloring. We give bounds on  $\chi'_s(G)$  in terms of the maximum degree  $\Delta(G)$  of a graph  $G$  when  $G$  is sparse, namely, when  $G$  is 2-degenerate or when the maximum average degree  $\text{Mad}(G)$  is small. We prove that the strong chromatic index of each 2-degenerate graph  $G$  is at most  $5\Delta(G) + 1$ . Furthermore, we show that for a graph  $G$ , if  $\text{Mad}(G) < 8/3$  and  $\Delta(G) \geq 9$ , then  $\chi'_s(G) \leq 3\Delta(G) - 3$  (the bound  $3\Delta(G) - 3$  is sharp) and if  $\text{Mad}(G) < 3$  and  $\Delta(G) \geq 7$ , then  $\chi'_s(G) \leq 3\Delta(G)$  (the restriction  $\text{Mad}(G) < 3$  is sharp).

## 1. INTRODUCTION

A *strong  $k$ -edge-coloring* of a graph  $G$  is a mapping from  $E(G)$  to  $\{1, 2, \dots, k\}$  such that every two adjacent edges or two edges adjacent to the same edge receive distinct colors. In other words, the graph induced by each color class is an induced matching. The *strong chromatic index* of  $G$ , denoted by  $\chi'_s(G)$ , is the smallest integer  $k$  such that  $G$  admits a strong  $k$ -edge-coloring.

Strong edge-coloring was introduced by Fouquet and Jolivet [12, 13] and was used to solve the frequency assignment problem in some radio networks. For more details on applications see [2, 21, 22, 23].

An obvious upper bound on  $\chi'_s(G)$  (given by a greedy coloring) is  $2\Delta(G)(\Delta(G) - 1) + 1$  where  $\Delta(G)$  denotes the maximum degree of  $G$ . Erdős and Nešetřil [9, 10] conjectured that for every graph  $G$  with maximum degree  $\Delta$ ,

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even} \\ \frac{5}{4}\Delta^2 - \frac{\Delta}{2} + \frac{1}{4} & \text{if } \Delta \text{ is odd} \end{cases}$$

The bounds in the conjecture are sharp, if the conjecture is true.

The first nontrivial upper bound on  $\chi'_s(G)$  was given by Molloy and Reed [20], who showed that  $\chi'_s(G) \leq 1.998\Delta^2$ , if  $\Delta$  is sufficiently large. Recently, the coefficient 1.998 was improved to 1.93 (again, for sufficiently large  $\Delta$ ) by Bruhn and Joos [5]. For  $\Delta = 3$ , the conjecture was settled independently by Andersen [1] and by Horák, Qing, and Trotter [15]. Cranston [8] proved that every graph with  $\Delta \leq 4$  admits a strong edge-coloring with 22 colors, which is by 2 more than the conjectured bound.

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The first author is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (NRF-2015R1C1A1A02036398). The second author is supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant Agreements no. 306349 (J. Kim). The research is conducted while the second author visited University of Illinois Urbana-Champaign as part of BRIDGE Strategic Partnership. Research of the third author is supported in part by NSF grant DMS-1600592 and by grants 15-01-05867 and 16-01-00499 of the Russian Foundation for Basic Research. Research of the fourth author received financial support from the French State, managed by the French National Research Agency (ANR) within the "Investments for the future" Programme IdEx Bordeaux - CPU (ANR-10-IDEX-03-02).

The strong chromatic index was studied for various families of graphs, such as cycles, trees,  $d$ -dimensional cubes, chordal graphs, and Kneser graphs, see [19]. There was also a series of papers [11, 14, 17] on strong edge-coloring planar graphs. In particular, Faudree, Gyárfás, Schelp, and Tuza [11] proved that  $\chi'_s(G) \leq 4\Delta + 4$  for every planar graph  $G$  with maximum degree  $\Delta$  and exhibited, for every integer  $\Delta \geq 2$ , a planar graph with maximum degree  $\Delta$  and strong chromatic index  $4\Delta - 4$ . Borodin and Ivanova [4] showed that every planar graph  $G$  with maximum degree  $\Delta \geq 3$  and girth  $g \geq 40\lfloor \Delta/2 \rfloor$  satisfies  $\chi'_s(G) \leq 2\Delta - 1$ , and that the bound  $2\Delta - 1$  is sharp. Chang, Montassier, Pecher, and Raspaud [6] relaxed the restriction on  $g$  to  $g \geq 10\Delta + 46$  for  $\Delta \geq 4$ .

Hudák, Lužar, Soták, and Škrekovski [16] proved that  $\chi'_s(G) \leq 3\Delta + 6$  for every planar graph  $G$  with maximum degree  $\Delta \geq 3$  and girth  $g \geq 6$ . Recently, Bensmail, Harutyunyan, Hocquard, and Valicov [3] improved the upper bound  $3\Delta + 6$  for such graphs to  $3\Delta + 1$ . With stronger restriction  $g \geq 7$ , Ruksasakchai and Wang [24] reduced the bound  $3\Delta + 1$  to  $3\Delta$ .

Clearly, planar graphs with large girth are sparse. The problem of strong edge-coloring was also studied for general sparse graphs. A natural measure of sparsity is *degeneracy*: a graph  $G$  is  $d$ -degenerate if every subgraph  $G'$  of  $G$  has a vertex of degree at most  $k$  (in  $G'$ ). Chang and Narayanan [7] proved that  $\chi'_s(G) \leq 10\Delta - 10$  for every 2-degenerate graph  $G$  with maximum degree  $\Delta \geq 2$ . Luo and Yu [18] improved the bound  $10\Delta - 10$  to  $8\Delta - 4$ . A more general bound by Yu [26] allowed to reduce the bound for 2-degenerate graphs to  $6\Delta - 5$ , and Wang [25] improved it to  $6\Delta - 7$ .

In this paper, we prove three bounds on the strong chromatic index of sparse graphs in terms of the maximum degree. Two of our bounds yield new bounds for planar graphs with girths 6 and 8.

Our first result is on 2-degenerate graphs. It improves the aforementioned bounds in [7, 18, 25] for  $\Delta \geq 9$ .

**Theorem 1.1.** *Every 2-degenerate graph  $G$  with maximum degree  $\Delta$  satisfies  $\chi'_s(G) \leq 5\Delta + 1$ .*

A finer measure of sparsity is the *maximum average degree*, denoted  $\text{Mad}(G)$ , which is the maximum of  $2\frac{|E(G')|}{|V(G')|}$  over all nontrivial subgraphs  $G'$  of a graph  $G$ . By definition,  $\text{Mad}(G) < 4$  for every 2-degenerate graph  $G$ . Two of our results show that if  $\text{Mad}(G) < 3$ , then we can use significantly fewer than  $5\Delta$  colors. The graphs  $K_\Delta(t)$  defined below show that our bounds are almost optimal. Let  $K_\Delta(t)$  be the graph obtained from  $K_t$  by adding  $\Delta - t + 1$  pendant edges to each vertex in  $K_t$ . It is easy to check that  $\text{Mad}(K(t)) = t - 1$  and  $\chi'_s(K_\Delta(t)) = |E(K_\Delta(t))| = t\Delta - \binom{t}{2}$ . In particular,

- $\text{Mad}(K_\Delta(2)) = 1$  and  $\chi'_s(K_\Delta(2)) = 2\Delta - 1$ ,
- $\text{Mad}(K_\Delta(3)) = 2$  and  $\chi'_s(K_\Delta(3)) = 3\Delta - 3$ ,
- $\text{Mad}(K_\Delta(4)) = 3$  and  $\chi'_s(K_\Delta(4)) = 4\Delta - 6$ .

Our second result is:

**Theorem 1.2.** *Every graph  $G$  with maximum average degree less than  $8/3$  and maximum degree  $\Delta \geq 9$  satisfies  $\chi'_s(G) \leq 3\Delta - 3$ .*

The graph  $K_\Delta(3)$  above shows that the bound  $3\Delta - 3$  is best possible. The graph  $K'_\Delta(4)$  defined below shows that the bound on the maximum average degree is close to optimal:

We start from a copy  $R$  of  $K_4$  with vertex set  $\{v_1, v_2, v_3, v_4\}$  and let  $K'_\Delta(4)$  be the graph obtained from  $R$  by subdividing the edge  $v_3v_4$  with a vertex  $u$  and then adding  $\Delta - 3$  pendant edges to each of  $v_1, v_2, v_3$ . It is not hard to check that the maximum degree of  $K'_\Delta(4)$  is  $\Delta$ ,  $\text{Mad}(K'_\Delta(4)) = 14/5$ , and  $\chi'_s(K'_\Delta(4)) = 3\Delta - 2$ .

Our last result is:

**Theorem 1.3.** *Every graph  $G$  with maximum average degree less than 3 and maximum degree  $\Delta \geq 7^1$  satisfies  $\chi'_s(G) \leq 3\Delta$ .*

<sup>1</sup>We can prove the result for  $\Delta \geq 6$ , but that would make the proof longer and more complicated.

Note that for small  $\Delta$ , namely for  $\Delta \leq 4$ , the slightly weaker bound of  $3\Delta + 1$  was proved by Ruksasakchai and Wang [24]. Since  $\text{Mad}(K_\Delta(4)) = 3$  and  $\chi'_s(K_\Delta(4)) = 4\Delta - 6$ , the restriction on the maximum average degree in Theorem 1.3 is best possible for  $\Delta \geq 7$ . The graph  $K'_\Delta(4)$  above with  $\text{Mad}(K'_\Delta(4)) = 14/5$  and  $\chi'_s(K'_\Delta(4)) = 3\Delta - 2$  shows that the bound  $3\Delta$  is also close to the best possible.

Since  $\text{Mad}(G) < \frac{2g}{g-2}$  for every planar graph  $G$  with girth  $g$ , Theorem 1.2 yields that  $\chi'_s(G) \leq 3\Delta - 3$  for every planar graph  $G$  with maximum degree  $\Delta \geq 9$  and girth  $g \geq 8$  and Theorem 1.3 implies that  $\chi'_s(G) \leq 3\Delta$  for every planar graph  $G$  with maximum degree  $\Delta \geq 7$  and girth  $g \geq 6$ . The last result improves the bounds in [3, 16, 24] mentioned above for  $\Delta \geq 7$ .

The structure of the paper is as follows. In the next section we introduce some notation and prove useful lemmas. In Section 3, 4, and 5, we prove Theorem 1.1, Theorem 1.2, and Theorem 1.3, respectively.

## 2. NOTATION AND PRELIMINARIES

Let  $[k] := \{1, \dots, k\}$ . For a function  $f$  defined on a set  $A'$  with  $A' \subseteq A$ , we denote  $f(A) := f(A')$ . For a graph  $G$ , let  $\bar{d}(G)$  be the average degree of  $G$ . We define  $\text{Mad}(G) := \max_{H \subseteq G} \bar{d}(H)$ . A vertex  $v \in V(G)$  is a  $d_G^+$ -vertex if  $d_G(v) \geq d$ . If  $G$  is clear from the context, then we simply say that that  $v$  is a  $d^+$ -vertex. A  $d_G^+$ -neighbor of a vertex  $v \in V(G)$  is a neighbor of  $v$  that is a  $d_G^+$ -vertex. A  $d_G^-$ -vertex, a  $d_G$ -vertex, a  $d_G^-$ -neighbor, and a  $d_G$ -neighbor are defined similarly.

For a vertex  $v \in V(G)$ , let  $N_G(v)$  denote the set of all neighbors of  $v$  in  $G$  and let  $\Gamma_G(v)$  denote the set of all edges incident to  $v$  in  $G$ . For an edge  $e = uv$ , let

$$N_G[e] := \Gamma_G(u) \cup \Gamma_G(v) \text{ and } N_G^2[e] := \bigcup_{w \in N_G(u) \cup N_G(v)} \Gamma_G(w).$$

A function  $f : E(G) \rightarrow [k]$  is a *strong  $k$ -edge-coloring* of  $G$  if  $f(e) \neq f(e')$  for any  $e, e' \in E(G)$  with  $e' \in N_G^2[e] \setminus \{e\}$ . Since below we only consider strong edge-colorings, for brevity we will simply call them *colorings*. A function  $f : E' \rightarrow [k]$  is a *partial  $k$ -coloring* of  $G$  on  $E'$  if  $E' \subseteq E(G)$  and  $f(e) \neq f(e')$  for any  $e, e' \in E(G')$  with  $e' \in N_G^2[e] \setminus \{e\}$ . For a partial  $k$ -coloring  $f : E' \rightarrow [k]$  of a graph  $G$  and  $e \in E(G)$ , let the  *$f$ -multiplicity of  $e$* ,  $m(f, e)$ , be  $m(f, e) := |N_G^2[e] \cap E'| - |f(N_G^2[e])|$ . Note that  $m(f, e)$  counts multiple occurrences of all colors in  $N_G^2[e]$ . By definition,  $|f(N_G^2[e])| = |N_G^2[e] \cap E'| - m(f, e)$ . In particular, if  $N_G^2[e]$  contains two edges with the same color, then  $m(f, e) \geq 1$ .

For a partial  $k$ -coloring  $f$  of  $G$ , then we say that a sequence  $(E_1, E_2, \dots, E_s)$  of pairwise disjoint subsets of  $E(G)$  is a  *$(f, k)$ -degenerate sequence for  $G$*  if the following holds:

- $f : E(G) \setminus (\bigcup_{i=1}^s E_i) \rightarrow [k]$  is a partial  $k$ -coloring of  $G$ .
- For every  $i \in [s]$  and  $e \in E_i$ ,  $|N_G^2[e] \setminus \bigcup_{j=i+1}^s E_j| \leq k + m(f, e)$ .

For a partial  $k$ -coloring  $f$  of a graph  $G$ , the graph  $G$  is  *$(f, k)$ -degenerate* if there exists a  $(f, k)$ -degenerate sequence  $(E_1, \dots, E_s)$  for  $G$ . If  $E_i = \{e_i\}$ , then for simplicity, instead of  $(E_1, \dots, E_s)$ , we write  $(E_1, \dots, E_{i-1}, e_i, E_{i+1}, \dots, E_s)$ .

The following lemma regarding degeneracy is useful.

**Lemma 2.1.** *If a graph  $G$  has a partial  $k$ -coloring  $f$  and is  $(f, k)$ -degenerate, then  $\chi'_s(G) \leq k$ .*

*Proof.* Assume we have a partial  $k$ -coloring  $f$  of  $G$  with domain  $E_0$  and  $(E_1, \dots, E_s)$  is a  $k$ -degenerate sequence on  $G$ . Let  $S = (e_1, \dots, e_t)$  be an ordering of edges in  $\bigcup_{i=1}^s E_i$  such that for  $j \in [s-1]$ , all edges in  $E_j$  come before any edge in  $E_{j+1}$ .

We iteratively color edges in  $S$  in order to extend  $f$  to  $f_1, \dots, f_t$ . Assume that we have colored  $e_1, \dots, e_{i-1}$  for  $i \in [t]$  and have a partial  $k$ -coloring  $f_{i-1}$  on  $E_0 \cup \{e_1, \dots, e_{i-1}\}$ . Let  $e_i \in E_j$ . Note that  $m(f, e) \leq m(f_{i-1}, e)$  for any  $e \in E(G)$ . Then since  $(E_1, \dots, E_s)$  is a  $(f, k)$ -degenerate sequence

for  $G$ ,

$$\begin{aligned} |f_{i-1}(N_G^2[e_i])| &= |N_G^2[e_i] \setminus \{e_i, \dots, e_t\}| - m(f_{i-1}, e_i) \leq \left| N_G^2[e_i] \setminus \left( \{e_i\} \cup \bigcup_{\ell=j+1}^s E_\ell \right) \right| - m(f, e_i) \\ &= \left| N_G^2[e_i] \setminus \bigcup_{\ell=j+1}^s E_\ell \right| - |\{e_i\}| - m(f, e_i) \leq k - 1. \end{aligned}$$

Thus we can choose a color  $c \in [k] \setminus f_{i-1}(N_G^2[e_i])$ . Let

$$f_i(e) := \begin{cases} f_{i-1}(e) & \text{if } e \in E_0 \cup \{e_1, \dots, e_{i-1}\}, \\ c & \text{if } e = e_i. \end{cases}$$

Now,  $f_i$  is a partial  $k$ -coloring of  $G$  with domain  $E_0 \cup \{e_1, \dots, e_i\}$ . By repeating this process, we get a strong  $k$ -edge-coloring of  $G$ . Thus  $\chi'_s(G) \leq k$ .  $\square$

We say a vertex  $u$  is *pale* in  $G$  if  $u$  has at most two  $3_G^+$ -neighbors. A vertex  $u$  is *light* in  $G$  if all vertices in  $N_G(u)$  except at most two are pale. Since a vertex with degree two is pale, each pale vertex is also light.

**Lemma 2.2.** *If  $G'$  is a subgraph of  $G$  with  $\delta(G') \geq 3$ , then  $G'$  has no vertex that is light in  $G$ .*

*Proof.* Assume  $G$  contains a subgraph  $G'$  with  $\delta(G') \geq 3$  and  $v \in V(G')$  is light in  $G$ . Then  $|N_{G'}(v)| \geq 3$ . So by the definition of “light”, there is a pale  $w \in N_{G'}(v)$ . Similarly,  $|N_{G'}(w)| \geq 3$ . So by the definition of “pale”, some  $u \in N_{G'}(w)$  is a  $2_G^-$ -vertex, contradicting  $\delta(G') \geq 3$ .  $\square$

**Lemma 2.3.** *Every 2-degenerate graph  $G$  with  $\Delta(G) \geq 3$  has a vertex  $v$  of degree at least three that is adjacent to at most two vertices of degree at least three.*

*Proof.* Let  $V'$  denote the set of  $3^+$ -vertices. Since  $\Delta(G) \geq 3$ , we know that  $V' \neq \emptyset$ . Since  $G$  is 2-degenerate,  $G[V']$  is also 2-degenerate. In particular,  $G[V']$  has a vertex  $v$  of degree at most 2. This  $v$  satisfies the lemma.  $\square$

### 3. 2-DEGENERATE GRAPHS

In this section we prove the following stronger result, which implies Theorem 1.1.

**Theorem 3.1.** *Let a 2-degenerate graph  $G$  and a positive integer  $D \geq 2$  satisfy the following:*

(A1)  $\Delta(G) \leq D + 2$ .

(A2) *For  $t \in [2]$ , every vertex  $v \in V(G)$  with  $d_G(v) = D + t$  is adjacent to at least  $t$  vertices of degree one.*

*Then  $\chi'_s(G) \leq 5D + 1$ .*

*Proof.* Let  $G$  be a counterexample to the theorem with the fewest  $2^+$ -vertices, and subject to this, with the fewest edges. Let  $G^*$  be obtained from  $G$  by deleting all vertices of degree 1 in  $G$ . By (A2),  $\Delta(G^*) \leq D$ .

First we prove that

$$\text{if } v \in V(G^*) \text{ and } d_{G^*}(v) \leq 2, \text{ then } d_{G^*}(v) = d_G(v). \quad (3.1)$$

Indeed, suppose to the contrary that  $N_{G^*}(v) = \{w_1, \dots, w_t\}$  where  $t \in [2]$  and there is  $u \in N_G(v) - N_{G^*}(v)$ . The graph  $G' := G - \{u\}$  is 2-degenerate and has no more  $2_{G'}^+$ -vertices than  $G$  has  $2_G^+$ -vertices. Furthermore,  $G'$  has strictly fewer edges than  $G$ . By the minimality of  $G$ , the graph  $G'$  has a  $(5D + 1)$ -coloring  $f$ . Since

$$|N_G^2[uv]| \leq \left| \Gamma_G(v) \cup \bigcup_{i=1}^t \Gamma_G(w_i) \right| \leq D + 2 + D + 2 + D + 2 - 2 \leq 5D + 1,$$

when  $D \geq 2$ , we can extend  $f$  to  $uv$ , a contradiction. This proves (3.1). In particular, (3.1) yields

$$\delta(G^*) \geq 2. \quad (3.2)$$

If  $\Delta(G^*) \leq 2$ , then by (3.2),  $G^*$  is a disjoint union of cycles. Then by (3.1),  $G$  itself is a disjoint union of cycles. So by the minimality of  $G$ , it is a cycle. Since each edge of a cycle has at most four edges at distance at most 2, it is 5-strong edge-colorable. This contradicts the choice of  $G$  since  $5 \leq 5D + 1$ . Thus  $\Delta(G^*) > 2$ . Then by Lemma 2.3,  $G^*$  has a vertex  $v$  with  $d_{G^*}(v) \geq 3$  that is adjacent to at most two  $3_{G^*}^+$ -vertices.

Let  $N_G(v) = \{v_1, v_2, u_1, \dots, u_t, u'_1, \dots, u'_s\}$ , where  $u_i \in V(G^*)$  and  $d_{G^*}(u_i) = 2$  for all  $i \in [t]$  and  $u'_j \in V(G) - V(G^*)$  for all  $j \in [s]$ . (It could be that  $d_{G^*}(v_1) = 2$  and/or  $d_{G^*}(v_2) = 2$ .) By the choice of  $v$ , we know  $t \geq 1$ . By the definition of  $G^*$ ,  $d_G(u'_j) = 1$  for each  $j \in [s]$ . By (3.1) and (3.2),  $d_{G^*}(u_i) = d_G(u_i) = 2$  for each  $i \in [t]$ . For each  $i \in [t]$ , let  $N_G(u_i) = \{v, w_i\}$ . By (A1),  $d_G(v) = t + s + 2 \leq D + 2$ . By (A2),  $t + 2 \leq D$ .

Suppose  $w_1$  has exactly  $h$  neighbors of degree 1 in  $G$ . Let  $H$  be the graph obtained from  $G - vu_1$  by adding  $\ell := \max\{0, 2 - h\}$  new vertices  $x_1, \dots, x_\ell$ , each of which is adjacent only to  $w_1$ . Since the degree of  $u_1$  in  $G$  is 2 and in  $H$  is 1,  $H$  has fewer  $2^+$ -vertices than  $G$ . Also by construction,

$$w_1 \text{ has at least 3 neighbors of degree 1 in } H, \text{ say } u_1, u'_1, \text{ and } u''_1. \quad (3.3)$$

It is not hard to check that  $H$  inherits properties (A1) and (A2) from  $G$ . Note that  $H$  has fewer  $2^+$ -vertices than  $G$ . So, by the minimality of  $G$ , the 2-degenerate graph  $H$  has a  $(5D + 1)$ -coloring  $f$ . By (3.3), we can switch the colors of  $w_1u_1, w_1u'_1$ , and  $w_1u''_1$  so that

$$f(w_1u_1) \notin \{f(vu_1), f(vu_2)\}. \quad (3.4)$$

**Case 1.**  $f(w_1u_1) \notin \{f(vu_2), \dots, f(vu_t)\} \cup \{f(vu'_1), \dots, f(vu'_s)\}$ . Together with (3.4), this yields that  $f|_{E(G)}$  is a partial coloring of  $G$  where only  $vu_1$  is not colored. Since

$$\begin{aligned} |N_G^2[vu_1]| &\leq d_G(v_1) + d_G(v_2) + (d_G(w_1) - 1) + s + \sum_{i=1}^t d_G(u_i) \\ &\leq (D + 2) + (D + 2) + (D + 1) + s + 2t \leq 3D + 5 + 2D - 4 \leq 5D + 1, \end{aligned} \quad (3.5)$$

we can extend  $f$  to  $vu_1$ .

**Case 2.** There is  $u \in \{u_2, \dots, u_t\} \cup \{u'_1, \dots, u'_s\}$  such that  $f(w_1u_1) = f(vu)$ . Then by (3.5), we can choose  $f(vu_1)$  so that the only conflict in  $f$  will be that  $f(w_1u_1) = f(vu)$ . Let  $f'$  be obtained from  $f$  by uncoloring  $vu$ . If  $u = u_i$  for some  $i \in [t] \setminus \{1\}$ , then we have Case 1 with  $f'$  in place of  $f$  and  $u_i$  in place of  $u_1$ . Otherwise,  $u = u'_j$  for some  $j \in [s]$ , and we can extend  $f'$  to  $vu'_j$ , since

$$|N_G^2[vu'_j]| \leq d_G(v_1) + d_G(v_2) + s + \sum_{i=1}^t d_G(u_i) \leq 2(D + 2) + 2t + s \leq 4D + 2 \leq 5D + 1.$$

This proves the theorem. □

#### 4. GRAPHS WITH MAXIMUM AVERAGE DEGREE LESS THAN $8/3$

In this section we prove Theorem 1.2:

*If  $G$  is a graph with  $\text{Mad}(G) < 8/3$  and  $\Delta(G) \geq 9$ , then  $\chi'_s(G) \leq 3\Delta(G) - 3$ .*

Similarly to the proof of Theorem 3.1, consider a counterexample  $G$  with the fewest  $2^+$ -vertices, and subject to this with the fewest edges. Let  $G^*$  be the graph obtained from  $G$  by deleting all vertices of degree 1. Let  $\Delta = \Delta(G)$ .

**Claim 4.1.**  $\delta(G^*) \geq 2$ .

*Proof.* Suppose  $u$  is a  $1_{G^*}$ -vertex where  $w$  is the neighbor of  $u$  in  $G^*$ . Then there exists  $v \in N_G(u) - w$  with  $d_G(v) = 1$ . By the minimality of  $G$ , the graph  $G - \{v\}$  has a  $(3\Delta - 3)$ -coloring  $f$ . Then  $f$  is a partial  $(3\Delta - 3)$ -coloring of  $G$ , and since

$$|N_G^2[uv]| = \sum_{x \in N_G(u)} d_G(x) \leq d_G(w) + |N_G(u) \setminus N_{G^*}(u)| \leq \Delta + (\Delta - 1) \leq 3\Delta - 3, \quad (4.1)$$

we can extend  $f$  to  $uv$ , a contradiction.  $\square$

Say that a vertex  $v$  is *special* if  $d_{G^*}(v) = 2$  and  $v$  is adjacent to a  $(\Delta - 1)_{G^*}^+$ -vertex.

**Claim 4.2.** *If  $u_1$  and  $u_2$  are two adjacent  $2_{G^*}$ -vertices, then both  $u_1$  and  $u_2$  are special.*

*Proof.* Let  $N_{G^*}(u_1) = \{w_1, u_2\}$  and  $N_{G^*}(u_2) = \{u_1, w_2\}$ . Assume  $u_1$  is not special, so that  $d_{G^*}(w_1) \leq \Delta - 2$ .

Obtain the graph  $H$  from  $G$  by first deleting all  $1_G$ -neighbors of  $u_1$  and  $u_2$ , then deleting the edge  $u_1u_2$ , and then adding leaves adjacent to  $w_1$  and  $w_2$  so that  $d_H(w_i) = \Delta$  for  $i \in [2]$ . By construction,  $H$  has fewer vertices of degree at least 2 than  $G$ . Also, either  $\text{Mad}(G) < 2$  and hence  $\text{Mad}(H) < 2$  or  $\text{Mad}(H) \leq \text{Mad}(G)$  and hence  $\text{Mad}(H) < 8/3$ . So  $H$  has a  $(3\Delta - 3)$ -coloring  $f$  by the minimality of  $G$ . Also  $w_1$  is incident to at least three pendant edges, one of which is  $u_1w_1$ . Let other two such edges be  $w_1v_1$  and  $w_1v_2$ .

By the definition of  $H$ ,  $|f(\Gamma_H(w_1))| = |f(\Gamma_H(w_2))| = \Delta$ . We define a special color  $\alpha$  as follows. If  $f(w_2u_2) \in f(\Gamma_H(w_1))$ , then let  $\alpha := f(w_2u_2)$ . Otherwise,  $f(\Gamma_H(w_1)) \setminus f(\Gamma_H(w_2)) \neq \emptyset$ , and we let  $\alpha$  be any color in this difference. After this, we switch the colors  $f(u_1w_1)$  and  $f(w_1v_1)$ , if necessary, so that  $f(u_1w_1) \neq \alpha$ . In particular, this means  $f(u_1w_1) \neq f(u_2w_2)$ , so now  $f|_{E(G)}$  is a partial coloring of  $G$ .

We extend  $f$  to  $u_1u_2$  by coloring it with a color not in  $f(\Gamma_H(w_1) \cup \Gamma_H(w_2))$ , which is nonempty since  $2\Delta + 1 \leq 3\Delta - 3$ . Now we will color the pendant edges of  $G$  incident with  $u_2$ , if there are any. In particular, if there is at least one such edge  $u_2u'_2$  and  $\alpha \neq f(w_2u_2)$ , then we start by letting  $f(u_2u'_2) = \alpha$ . This provides that

$$d_G(u_2) = 2 \text{ or } N_G^2[u_1z] \text{ has two edges of color } \alpha \text{ for each } 1_G\text{-neighbor } z \text{ of } u_1. \quad (4.2)$$

Then we color the remaining edges in  $\Gamma_G(u_2)$  one by one, since the only colored edges that could be in conflict are in  $\Gamma_G(w_2) \cup \Gamma_G(u_2) \cup \{w_1u_1\}$ , and there are at most  $2\Delta + 1$  such edges. Finally, we remove the edges in  $E(H) - E(G)$  and color the pendant edges of  $G$  incident with  $u_1$  one by one. For every pendant edge  $u_1z \in E(G)$  incident with  $u_1$ , at the moment of coloring  $u_1z$ , the number  $M(u_1z)$  of the colors forbidden for  $u_1z$  is at most  $|N_G^2[u_1z] - \{u_1z\}|$ . Moreover, by (4.2), if  $d_G(u_2) \geq 3$ , then  $M(u_1z) \leq |N_G^2[u_1z] - \{u_1z\}| - 1$ . In any case,

$$M(u_1z) \leq d_G(w_1) + \max\{2, d_G(u_2) - 1\} + (d_G(u_1) - 3) \leq \Delta + \max\{2, \Delta - 1\} + (\Delta - 3) = 3\Delta - 4.$$

So we can always find a free color for  $u_1z$ . This yields a  $(3\Delta - 3)$ -coloring of  $G$ , a contradiction. This shows that  $u_1$  is special, and by symmetry, this also shows that  $u_2$  is special.  $\square$

**Claim 4.3.** *If  $u$  is a  $3_{G^*}$ -vertex with two  $2_{G^*}$ -neighbors  $v_1$  and  $v_2$ , then at least one of  $v_1, v_2$  is special.*

*Proof.* Let  $N_{G^*}(v_i) = \{u, v'_i\}$  for  $i \in [2]$  and let  $N_{G^*}(u) = \{w, v_1, v_2\}$ . Suppose to the contrary that both  $v_1, v_2$  are not special, so  $v'_1, v'_2$  are both  $(\Delta - 2)_{G^*}^-$ -vertices. We construct a graph  $H$  from  $G - \{uv_1, uv_2\}$  by deleting all  $1_G$ -neighbors of  $u, v_1$ , and  $v_2$  and then adding leaves to  $v'_1, v'_2$ , and  $w$  to make the degrees of  $v'_1, v'_2$ , and  $w$  equal to  $\Delta$ . Since  $\text{Mad}(G) < 8/3$ , and we were adding only 1-vertices,  $\text{Mad}(H) < 8/3$ . Since  $u, v_1$ , and  $v_2$  are leaves in  $H$  but not in  $G$ , the graph  $H$  has fewer  $2^+$ -vertices than  $G$ . Thus by the minimality of  $G$ , the graph  $H$  has a  $(3\Delta - 3)$ -coloring  $f$ .

For  $i \in [k]$ , since  $v'_i$  is a  $(\Delta - 2)_{G^*}^-$ -vertex, it is adjacent to at least three leaves in  $H$ , including  $v_i$ ; let  $v'_{i,1}$  and  $v'_{i,2}$  be the other leaves adjacent to  $v'_i$  in  $H$ .



We now define special colors  $\alpha_1$  and  $\alpha_2$  as follows. For  $i \in [2]$ , if  $f(uw) \in f(\Gamma_H(v'_i))$ , then we let  $\alpha_i := f(uw)$ . Otherwise, the set  $f(\Gamma_H(v'_i)) - f(\Gamma_H(w))$  is nonempty, and we let  $\alpha_i$  be any color in this difference. Note that  $\alpha_2 = \alpha_1$  is possible. By definition,

$$|\{\alpha_1, \alpha_2, f(uw)\} \cap f(\Gamma_H(v'_i))| \leq 2 \quad \text{for } i \in [2]. \quad (4.3)$$

Also, for  $i \in [2]$ , the colors  $f(v'_i v_i), f(v'_i v'_{i,1}), f(v'_i v'_{i,2})$  are all distinct. So by (4.3), at least one of them is not in  $\{\alpha_1, \alpha_2, f(uw)\}$ . Hence we can switch these colors so that  $f(v'_i v_i) \notin \{\alpha_1, \alpha_2, f(uw)\}$ . Then  $f|_{E(G)}$  is a partial  $(3\Delta - 3)$ -coloring of  $G$ .

Let  $H' = H + v_1 u + v_2 u$ . Then

$$|N_{H'}^2[uv_i]| \leq |\Gamma_{H'}(w)| + |\Gamma_{H'}(v'_i)| + |\Gamma_{H'}(v_{3-i})| + 1 \leq \Delta + \Delta + 2 + 1 \leq 3\Delta - 3.$$

Thus for  $i \in [2]$ , we can extend  $f$  to  $uv_i$  by coloring it with a color not in  $f(N_{H'}^2[uv_i])$ .

Now we will color the pendant edges of  $G$  incident with  $u$ , if there are any. In particular, if there are at least two such edges  $uu'_1, uu'_2$ , and  $\alpha_1 \neq f(uu)$ , then we start by letting  $f(uu'_1) = \alpha_1$ , and then, if  $\alpha_2 \notin \{f(uw), \alpha_1\}$ , then  $f(uu'_2) = \alpha_2$ . This provides that for  $i \in [2]$ ,

$$d_G(u) \leq 4 \text{ or } N_G^2[v_i z] \text{ contains two edges of color } \alpha_i \text{ for every } 1_G\text{-neighbor } z \text{ of } v_i. \quad (4.4)$$

Then we color the remaining edges in  $\Gamma_G(u)$  one by one, since the only colored edges that could be in conflict are in  $\Gamma_G(w) \cup \Gamma_G(u) \cup \{v_1 v'_1, v_2 v'_2\}$ , and there are at most  $2\Delta + 1$  such vertices. Finally, we remove the edges in  $E(H) - E(G)$  and for  $i \in [2]$  color the leaves of  $G$  incident with  $v_i$  one by one. For every leaf  $v_i z \in E(G)$  incident with  $v_i$ , at the moment of coloring  $v_i z$ , the number  $M(v_i z)$  of the colors forbidden for  $v_i z$  is at most  $|N_G^2[v_i z] - \{v_i z\}|$ . Moreover, by (4.4), if  $d_G(u) \geq 5$ , then  $M(v_i z) \leq |N_G^2[v_i z] - \{v_i z\}| - 1$ . In any case,

$$M(v_i z) \leq d_G(v'_i) + \max\{4, d_G(u) - 1\} + (d_G(v_i) - 3) \leq \Delta + \max\{4, \Delta - 1\} + (\Delta - 3) = 3\Delta - 4.$$

So we can always find a free color for  $v_i z$ , for each  $i \in [2]$ . This yields a  $(3\Delta - 3)$ -coloring of  $G$ , a contradiction.  $\square$

Now we will complete the proof of the theorem using discharging: For each  $v \in V(G^*)$ , we let the *initial charge*  $\text{ch}(v) := d_{G^*}(v)$ , and then will move charge among vertices so that the *final charge*  $\text{ch}^*(v)$  is at least  $8/3$  for each  $v \in V(G^*)$ , but the total sum of charge will be preserved during the entire process. This will imply that

$$\sum_{v \in V(G^*)} d_{G^*}(v) = \sum_{v \in V(G^*)} \text{ch}(v) = \sum_{v \in V(G^*)} \text{ch}^*(v) \geq \frac{8}{3} |V(G^*)|, \quad (4.5)$$

contradicting the fact that  $\text{Mad}(G^*) < \frac{8}{3}$ .

The rules of discharging are the following.

- (R1) Each  $(\Delta - 1)_{G^*}^+$ -vertex sends charge  $2/3$  to each neighbor.
- (R2) Each vertex  $v$  with  $4 \leq d_{G^*}(v) \leq \Delta - 2$  sends charge  $1/3$  to every  $2_{G^*}$ -neighbor.
- (R3) Each  $3_{G^*}$ -vertex sends charge  $1/3$  to every  $2_{G^*}$ -neighbor that is not special.

By (4.5), the theorem will follow from the following claim:

**Claim 4.4.** *For every  $v \in V(G^*)$ ,  $\text{ch}^*(v) \geq \frac{8}{3}$ .*

*Proof.* We consider several cases depending on the degree of  $v$ .

If  $v$  is a special  $2_{G^*}$ -vertex, then it receives charge  $2/3$  from its  $(\Delta - 1)_{G^*}^+$ -neighbor by Rule (R1) and gives out nothing. Thus  $\text{ch}^*(v) \geq \text{ch}(v) + 2/3 = 8/3$ .

If  $v$  is a  $2_{G^*}$ -vertex but not special, then by Claim 4.2 it has two  $3_{G^*}^+$ -neighbors, and each of them sends  $v$  charge  $1/3$  either by (R2) or by (R3). Thus  $\text{ch}^*(v) \geq \text{ch}(v) + 1/3 + 1/3 = 8/3$ .

If  $d_{G^*}(v) = 3$  and  $v$  is adjacent to exactly  $t$  vertices of degree two, then by Claim 4.3, at least  $t - 1$  of them are special. Thus by Rule (R3),  $v$  sends out charge  $1/3$  to at most one of its neighbors. Hence  $\text{ch}^*(v) \geq \text{ch}(v) - 1/3 = 8/3$ .

If  $4 \leq d_{G^*}(v) \leq \Delta - 2$ , then by (R2),  $v$  sends charge at most  $1/3$  to each of its neighbors. Thus  $\text{ch}^*(v) \geq \text{ch}(v) - \frac{d_{G^*}(v)}{3} \geq 2\text{ch}(v)/3 \geq 8/3$ .

Finally, if  $d_{G^*}(v) \geq \Delta - 1$ , then by (R1),  $v$  sends charge  $2/3$  to each of its neighbors. Thus  $\text{ch}^*(v) \geq \text{ch}(v) - \frac{2d_{G^*}(v)}{3} = \text{ch}(v)/3 \geq (\Delta - 1)/3 \geq 8/3$  since  $\Delta \geq 9$ .  $\square$

## 5. GRAPHS WITH MAXIMUM AVERAGE DEGREE LESS THAN THREE.

In this section, instead of Theorem 1.3 we prove the following stronger result.

**Theorem 5.1.** *Let  $\Delta \geq 7$  be an integer and let  $G$  be a graph with no 3-regular subgraph. If  $\Delta(G) \leq \Delta$  and  $\text{Mad}(G) \leq 3$ , then  $\chi'_s(G) \leq 3\Delta(G)$ .*

**5.1. Set-up of the proof and some notation.** To prove Theorem 5.1, we consider a counterexample  $G$  with the fewest  $2^+$ -vertices, and subject to this with the fewest edges. Let  $G^*$  be the graph obtained from  $G$  by deleting all vertices of degree 1. Similarly to the proof of Theorem 1.2, we will show that vertices of “low” degree in  $G^*$  have neighbors with “high” degree, and based on this use discharging to prove that the average degree of  $G^*$  is greater than 3.

A feature not used in the previous proofs is the notion of potentials. For a graph  $G$  and  $A \subseteq V(G)$ , the *potential* of  $A$ , denoted  $\rho_G(A)$ , is defined as

$$\rho_G(A) := 3|A| - 2|E(G[A])|.$$

By definition,  $\text{Mad}(G) \leq 3$  if and only if  $\rho_G(A) \geq 0$  for all  $A \subseteq V(G)$ .

The following fact on potentials is easy to check.

**Lemma 5.2.** *For a graph  $G$  and any disjoint  $A, B \subset V(G)$ ,  $\rho_G(A) + \rho_G(B) = \rho_G(A \cup B) + \rho_G(A \cap B) + 2|E_G(A \setminus B, B \setminus A)|$ .*

A  $3_{G^*}$ -vertex is *poor* if it has exactly one  $2_{G^*}$ -neighbor. For a poor  $3_{G^*}$ -vertex  $u$ , an edge  $uw \in E(G^*)$  is the  *$u$ -sink* if  $d_{G^*}(w) = 2$  and  $N_{G^*}(u) - w$  contains a vertex  $u'$  with  $d_G(u') < \Delta$ . An edge is a *sink* if it is a  $u$ -sink for some poor  $3_{G^*}$ -vertex  $u$ . By definition, a poor  $3_{G^*}$ -vertex that is adjacent to two  $\Delta_{G^*}$ -vertices is not incident to a sink.

A  $2_{G^*}$ -vertex  $u$  is *very poor*, a vertex  $v$  is the *sponsor* of  $u$  and  $w$  is the *rival* of  $u$  if  $N_{G^*}(u) = \{v, w\}$  and one of the following holds:

- (T1)  $w$  is a  $2_{G^*}$ -vertex, or
- (T2)  $w$  is a  $3_{G^*}$ -vertex with two  $2_{G^*}$ -neighbors in  $G^*$  (including  $u$ ), or
- (T3)  $w$  is a  $4_{G^*}$ -vertex and each vertex in  $N_{G^*}(w)$  except at most one is either a  $2_{G^*}$ -vertex or a poor  $3_{G^*}$ -vertex.

Each  $2_{G^*}$ -vertex  $u$  that is not very poor is *poor*. See Figure 1 for an illustration. Poor vertices will be the recipients of charge in the discharging procedure.

If  $u$  is a very poor vertex with sponsor  $v$  and rival  $w$ , then edge  $uw$  is a *lower link* of  $u$ . Furthermore, if  $w$  is the rival of type (T3) of  $u$  and  $w'$  is a poor  $3_{G^*}$ -vertex in  $N_{G^*}(w)$ , then  $ww'$  a *semi-link* of  $w$ .

Let  $B(G)$  denote the set of all sinks in  $G$ . Similarly, let  $S(G)$  denote the set of all lower links in  $G$ , respectively. Let  $S'(G)$  denote the set of all semi-links in  $G$ . By definition, all poor vertices and all very poor vertices are light. Also the rival of each very poor vertex is light.

**5.2. Structure of  $G$  and  $G^*$ .** The next claim is in the spirit of Claims 4.2 and 4.3.

**Claim 5.3.** *Let  $v \in V(G^*)$ .*

(i) *If*

$$\sum_{u \in N_{G^*}(v)} d_G(u) \leq 2\Delta + d_{G^*}(v), \tag{5.1}$$

*then  $d_G(v) = d_{G^*}(v)$ .*



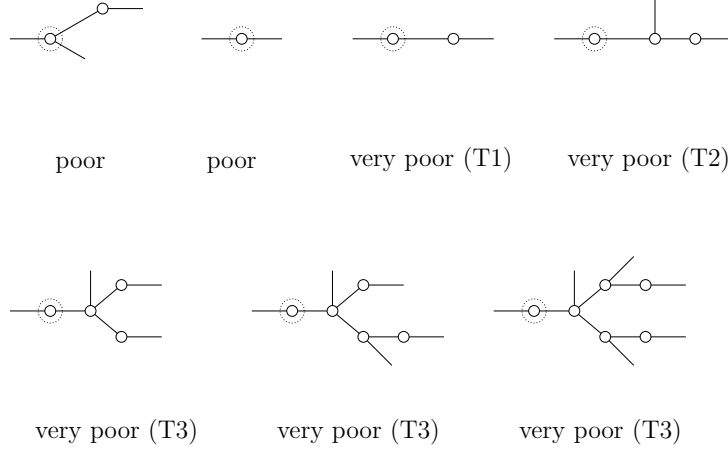


FIGURE 1. Poor vertices and very poor vertices.

(ii)  $d_{G^*}(v) \geq 2$ .

(iii) If  $d_{G^*}(v) = 2$ , then  $d_G(v) = 2$ .

(iv) If  $d_{G^*}(v) = 3$  and  $v$  has a neighbor  $u \in V(G^*)$  with  $d_G(u) \leq 3$ , then  $d_G(v) = 3$ . In particular, if  $v$  is poor  $3_{G^*}$ -vertex, then  $d_G(v) = 3$ .

By (ii),  $\delta(G^*) \geq 2$ .

*Proof.* (i) Suppose (5.1) holds. If  $d_G(v) \neq d_{G^*}(v)$ , then  $v$  has a  $1_G$ -neighbor  $w$ . By the minimality of  $G$ , the graph  $G - w$  has a  $3\Delta$ -coloring  $f$ , which is a partial  $3\Delta$ -coloring of  $G$ . However by (5.1),

$$|N_G^2[vw]| \leq d_G(v) + \sum_{u \in N_G(v)} (d_G(u) - 1) = d_G(v) + \sum_{u \in N_{G^*}(v)} d_G(u) - d_{G^*}(v) \leq d_G(v) + 2\Delta \leq 3\Delta.$$

Thus we can extend  $f$  to  $vw$ , a contradiction to the definition of  $G$ . This proves (i).

(ii) Assume that  $G^*$  has a  $1_{G^*}$ -vertex  $v$ . Let  $u$  be the unique neighbor of  $v$  in  $G^*$ . Since  $v \in V(G^*)$ , we have  $d_G(v) \geq 2 > d_{G^*}(v)$ . However,  $\sum_{w \in N_{G^*}(v)} d_G(w) = d_G(u) \leq \Delta$ , and (5.1) holds. Together with  $d_G(u) > d_{G^*}(u)$ , this contradicts (i). This proves (ii), which implies  $d_{G^*}(v) \geq 2$ .

(iii) If  $d_{G^*}(v) = 2$ , then  $\sum_{u \in N_{G^*}(v)} d_G(u) \leq 2\Delta$ . Thus (i) implies (iii).

(iv) If  $d_{G^*}(v) = 3$  and  $v$  has a neighbor  $u \in V(G^*)$  with  $d_G(u) \leq 3$ , then

$$\sum_{w \in N_{G^*}(v)} d_G(w) \leq (d_{G^*}(v) - 1)\Delta + d_G(u) \leq 2\Delta + 3.$$

Thus (5.1) holds, and (i) implies that  $d_G(v) = d_{G^*}(v) = 3$ . The ‘‘In particular’’ part follows from (ii) and the definition of a poor  $3_{G^*}$ -vertex.  $\square$

The next claim collects more properties of neighbors of poor and very poor vertices.

**Claim 5.4.** *Graph  $G$  possesses the following properties.*

(i) If  $u$  is either a poor or a very poor vertex and  $uv$  is a lower link, a semi-link, or a sink, then  $d_G(v) = d_{G^*}(v)$ .

(ii) If  $e$  is a sink, then  $|N_G^2[e]| \leq 3\Delta$ .

If  $e$  is a lower link, then  $|N_G^2[e] \setminus B(G)| \leq 3\Delta - 1$ .

If  $e$  is a semi-link, then  $|N_G^2[e] \setminus (B(G) \cup S(G))| \leq 3\Delta - 1$ .

- (iii) If  $v$  is a sponsor of a very poor vertex  $u$ , then  $v$  is a  $\Delta_{G^*}$ -vertex.
- (iv) If  $u$  is a poor  $3_{G^*}$ -vertex, then it is adjacent to at least one  $4_{G^*}^+$ -vertex.
- (v) If  $w$  is a rival of a very poor vertex  $u$ , then all but at most one neighbor of  $w$  is pale.

*Proof.* (i) Claim 5.3 (iii) and (iv) implies that  $d_G(u) = d_{G^*}(u)$ . If  $d_{G^*}(v) \leq 3$ , then Claim 5.3 (iv) implies  $d_G(v) = d_{G^*}(v)$ . Otherwise,  $u$  is either a very poor  $2_{G^*}$ -vertex of type (T3) or a poor  $3_{G^*}$ -vertex where  $v$  is a rival of a very poor vertex of type (T3). In any case, this implies  $d_{G^*}(v) = 4$ . Assume  $N_{G^*}(v) = \{u, u', u'', z\}$  where each of  $u', u''$  is either a very poor  $2_{G^*}$ -vertex or a poor  $3_{G^*}$ -vertex. Again Claim 5.3 (iii) and (iv) implies that  $d_G(p) = d_{G^*}(p)$  for  $p \in \{u, u', u''\}$ . Then  $d_G(z) + d_G(u) + d_G(u') + d_G(u'') \leq \Delta + 2 + 3 + 3 \leq 2\Delta + d_{G^*}(v)$ , in other words, (5.1) holds. Thus Claim 5.3 (i) implies that  $d_G(v) = d_{G^*}(v)$ .

(ii) Assume  $e = uv$  is a  $u$ -sink. This means  $u$  is a  $3_{G^*}$ -vertex,  $v$  is a  $2_{G^*}$ -vertex, and at least one vertex in  $N_{G^*}(u) \setminus \{v\}$  has degree less than  $\Delta$  in  $G$ . Then  $d_G(u) = d_{G^*}(u) = 3$  by Claim 5.3 (iv). Thus

$$|N_G^2[e]| \leq \sum_{w \in N_{G^*}(u) \setminus \{v\}} d_G(w) + \sum_{w \in N_{G^*}(v) \setminus \{u\}} d_G(w) + |\{e\}| \leq 2\Delta - 1 + \Delta + 1 \leq 3\Delta.$$

Assume now  $e = uv$  is a lower link of  $u$ . Then by definition and (i),  $d_G(u) = 2$  and  $2 \leq d_G(v) \leq 4$ . If  $d_G(v) = 4$ , let  $t$  be the number of poor  $3_{G^*}$ -vertices in  $N_G(v)$ ; then  $0 \leq t \leq 2$ . Depending on the type (T1)–(T3) of  $v$  we have

$$|N_G^2[uv]| \leq 1 + \sum_{y \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}} d_G(y) \leq \begin{cases} 2\Delta + 1 & \text{in case of (T1),} \\ 2\Delta + 3 & \text{in case of (T2),} \\ 2\Delta + 5 + t & \text{in case of (T3).} \end{cases}$$

Since  $d_{G^*}(v) \leq 4 < \Delta$ , each poor  $3_{G^*}$ -vertex  $w$  in  $N_{G^*}(v)$  is incident to a sink. Thus

$$|N_G^2[uv] \setminus B(G)| \leq \begin{cases} 2\Delta + 1 & \leq 3\Delta - 1 & \text{in case of (T1),} \\ 2\Delta + 3 & \leq 3\Delta - 1 & \text{in case of (T2),} \\ 2\Delta + 5 + t - t & \leq 3\Delta - 1 & \text{in case of (T3).} \end{cases}$$

If  $e = uv$  is a semi-link of  $v$ , then, by definition,  $v$  is the rival of some very poor vertex  $w$  and  $u$  is a poor  $3_{G^*}$ -vertex. So by (i),  $d_G(v) = d_{G^*}(v) = 4$ . Then each poor  $3_{G^*}$ -vertex  $u' \in N_{G^*}(v)$  is incident to a sink, since  $v \in N_{G^*}(u')$  satisfies  $d_G(v) < \Delta$ . Let  $t$  be the number of poor  $3_{G^*}$ -vertices in  $N_{G^*}(v)$ .

$$|N_G^2[uv]| \leq 1 + \sum_{y \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}} d_G(y) \leq 2\Delta + 2(3 - t) + 3t.$$

Since  $N_G^2[uv]$  contains at least  $t$  sinks and  $3 - t$  lower links,  $|N_G^2[uv] \cap (B(G) \cup S(G))| \geq 3$ . Hence  $|N_G^2[uv] \setminus (B(G) \cup S(G))| \leq 2\Delta + 6 + t - 3 \leq 3\Delta - 1$  since  $\Delta \geq 7$  and  $1 \leq t \leq 3$ .

(iii) Suppose  $v$  is a sponsor of a very poor vertex  $u$  and  $d_{G^*}(v) \leq \Delta - 1$ . Let  $w$  be the rival of  $u$ . Note that in each of the cases (T1)–(T3),  $\Gamma_G(w) \setminus (S(G) \cup S'(G))$  contains at most one edge; let this edge be  $e'$ . Let  $H$  be the graph obtained from  $G - uw$  by adding, if necessary, leaves adjacent only to  $v$  so that  $d_H(v) = \Delta$ . Then  $d_H(u) = 1$  and since  $d_{G^*}(v) \leq \Delta - 1$ ,  $v$  has a  $1_H$ -neighbor  $x$  distinct from  $u$ .

Adding leaves to a graph does not increase the maximum average degree, if it is at least 2. It also does not create new 3-regular subgraph. So  $\text{Mad}(H) \leq 3$  and  $H$  has no 3-regular subgraphs. Since  $H$  has fewer  $2^+$ -vertices than  $G$ , it has a  $3\Delta$ -coloring  $f'$ . Since  $x$  and  $u$  are symmetric in  $H$  and  $f'(vx) \neq f'(vu)$ , we may assume that  $f'(vu) \neq f'(e')$  by changing colors of  $vx$  and  $vu$  if necessary.

Let  $f(e) := f'(e)$  for each edge  $e \in E(H) \setminus (S(G) \cup S'(G) \cup B(G))$ . Then  $f|_{E(G)}$  is a partial  $3\Delta$ -coloring of  $G$  since  $f'(vu) \neq f'(e')$ . By (ii) and the fact that  $uw \in S'(G) \cup S(G) \cup B(G)$ ,

$(S'(G), S(G), B(G))$  is a  $(f|_{E(G)}, 3\Delta)$ -degenerate sequence for  $G$ . Thus we conclude that  $G$  is  $(f|_{E(G)}, 3\Delta)$ -degenerate. Thus Lemma 2.1 implies that  $\chi'_s(G) \leq 3\Delta$ , a contradiction. This proves (iii).

(iv) Suppose that the neighbors of a poor  $3_{G^*}$ -vertex  $u$  are  $v_1, v_2$ , and  $v_3$ , and  $d_{G^*}(v_i) \leq 3$  for  $i \in [3]$ . By the definition of a poor  $3_{G^*}$ -vertex, we may assume that  $d_{G^*}(v_1) = 2$  and  $d_{G^*}(v_2) = d_{G^*}(v_3) = 3$ . By Claim 5.3,  $d_G(w) = d_{G^*}(w)$  for  $w \in \{u, v_1, v_2, v_3\}$ . Consider  $H := G \setminus \{u\}$ , which has fewer vertices of degree at least 2 than  $G$ . The minimality of  $G$  implies that  $H$  has a  $3\Delta$ -coloring  $f$ . By the construction of  $H$ ,  $f$  is a partial  $3\Delta$ -coloring of  $G$ . Note that  $|N_G^2[uv_i]| \leq 2\Delta + 6 \leq 3\Delta$  for  $i \in [3]$ . Thus  $(uv_1, uv_2, uv_3)$  is a  $(f, 3\Delta)$ -degenerate sequence for  $G$ , and so  $G$  is  $(f, 3\Delta)$ -degenerate. So Lemma 2.1 implies that  $\chi'_s(H) \leq 3\Delta$ , a contradiction. This proves (iv).

(v) By definition, a poor  $3_{G^*}$ -vertex and a  $2_{G^*}$ -vertex are pale. Since each neighbor of  $w$  possibly except one is either a poor  $3_{G^*}$ -vertex or a  $2_{G^*}$ -vertex, (v) follows.  $\square$

**Claim 5.5.** *No vertex is a sponsor of two distinct very poor vertices.*

*Proof.* Assume that a vertex  $v$  is a sponsor of distinct very poor vertices  $u$  and  $u'$ . By Claim 5.3(iii), both  $u$  and  $u'$  are  $2_G$ -vertices. Let  $N_{G^*}(u) = \{v, w\}$  and  $N_{G^*}(u') = \{v, w'\}$ . By Claim 5.4(iii),  $v$  is a  $\Delta_{G^*}$ -vertex.

**Case 1.**  $vw \in E(G)$  (this includes the case  $w = u'$ ). Consider the graph  $H := G - \{uw\}$ . Since  $H$  is a proper subgraph of  $G$ , it satisfies the conditions of Theorem 5.1 and contains fewer  $2^+$ -vertices than  $G$ . So by the minimality of  $G$ , the graph  $H$  has a  $3\Delta$ -coloring  $f$ . Since  $vw \in E(G)$ ,  $f$  is a partial  $3\Delta$ -coloring of  $G$ , where only  $uw$  is not colored. By the definition of very poor vertices,  $d_{G^*}(w) \leq 4$  and  $v$  is the only possible neighbor of  $w$  that is neither poor nor very poor. Hence by Claims 5.3 and 5.4(i),

$$|N_G^2[ww]| \leq 1 + d_G(v) + \sum_{x \in N_G(w) - \{v, u\}} d_G(x) \leq 1 + \Delta + 2(3) < 3\Delta.$$

Thus we can extend  $f$  to  $uw$ , a contradiction.

**Case 2.**  $vw \notin E(G)$  and  $w = w'$ . Consider the graph  $H$  obtained from  $G - u$  by deleting the edge  $u'w$  and adding the edge  $vw$ . Then  $H$  has fewer  $2^+$ -vertices than  $G$ . Suppose  $V(H)$  contains a set  $A$  with  $\rho_H(A) < 0$ . Since  $\rho_G(A) \geq 0$ ,  $wv \in E(H[A])$ , so  $w, v \in V(H)$ . Also we may assume that  $u' \notin A$ , since  $\rho_H(A - u') \leq \rho_H(A)$ . However, since each of  $u$  and  $u'$  is adjacent to each of  $v$  and  $w$ , the graph  $G[A \cup \{u, u'\}]$  has 4 more edges than  $G[A]$ . So

$$\rho_G(A \cup \{u, u'\}) = \rho_G(A) + 2(3) - 4(2) = (\rho_H(A) + 2) + 6 - 8 = \rho_H(A) < 0,$$

a contradiction. Thus  $\rho_H(B) \geq 0$  for any  $B \subseteq V(H)$ . Similarly, if  $H$  contains a 3-regular subgraph  $H'$ , then  $H'$  contains both  $v$  and  $w$ . This means  $w$  has two neighbors in  $G$  that are not pale. This contradicts Claim 5.4 (v). Thus Lemma 2.2 implies that  $H$  has no 3-regular subgraphs containing  $w$ . Hence  $H$  has no 3-regular subgraphs at all. So  $H$  satisfies the conditions of Theorem 5.1 and by the minimality of  $G$ ,  $H$  has a  $3\Delta$ -coloring  $f'$ . Let

$$f(e) := \begin{cases} f'(e) & \text{if } e \in E(G) \cap E(H), \\ f'(vw) & \text{if } e = uv. \end{cases}$$

Since  $vw \in E(H)$ , colors  $f(uv)$  and  $f(u'v)$  are disjoint from  $f(\Gamma_G(w))$ . So  $f|_{E(G)}$  is a partial  $3\Delta$ -coloring of  $G$  and the only non-colored edges are  $wu$  and  $wu'$ . Similarly to the end of the proof

of Case 1,  $d_{G^*}(w) \leq 4$  and at most one neighbor of  $w$ , is neither poor nor very poor. Hence by Claims 5.3 and 5.4(i), for  $y \in \{u, u'\}$

$$|N_G^2[yw]| \leq 2 + d_G(v) + \sum_{x \in N_G(w) - \{u', u\}} d_G(x) \leq 2 + \Delta + \Delta + 3 < 3\Delta.$$

Thus we can extend  $f$  to  $uw$  and  $u'w$ , a contradiction.

**Case 3.**  $|\{u, u', w, w'\}| = 4$  and  $vw, vw' \notin E(G)$ . Since every rival of a very poor vertex is incident to at most one edge that is not in  $S(G) \cup S'(G)$ , let  $e'$  be the unique edge incident to  $w'$  such that  $e' \notin S(G) \cup S'(G)$ , if it exists. Similarly to Case 2, consider the graph  $H_1$  obtained from  $G - u$  by deleting the edge  $u'w'$  and adding the edge  $vw$ .

If  $H_1$  has a  $3\Delta$ -coloring  $f_1$ , then we may assume that  $\{f_1(vw), f_1(vu')\} = \{\alpha, \beta\}$  with  $\alpha \neq \beta$  and  $f_1(e') \neq \beta$ . Let

$$f(e) := \begin{cases} f_1(e) & \text{if } e \in E(G) \setminus (S(G) \cup S'(G) \cup B(G) \cup \{uv, vu'\}) \\ \alpha & \text{if } e = uv \\ \beta & \text{if } e = vu' \end{cases}$$

Then  $e'$  is the only edge in  $\Gamma_{H_1}(w')$  that is colored by  $f$ . Also  $\alpha, \beta \notin f(\Gamma_{H_1}(w))$ , since (due to the edge  $vw$ ) every edge in  $\Gamma_{H_1}(w)$  is distance at most one from  $vu'$  and  $vw$  in  $H_1$ . Thus  $f$  is a partial  $3\Delta$ -coloring of  $G$ , and uncolored edges are exactly the edges in  $S(G) \cup S'(G) \cup B(G)$ . (Note that  $wu, w'u' \in S(G)$ .) Hence Claim 5.4 (ii) implies that  $(S'(G), S(G), B(G))$  is a  $(f, 3\Delta)$ -degenerate sequence for  $G$ ; thus  $G$  is  $(f, 3\Delta)$ -degenerate, a contradiction.

Since  $H_1$  has fewer  $2^+$ -vertices than  $G$ , this means that  $H_1$  does not satisfy the conditions of our theorem. This means that either there exists a set  $A \subseteq V(H_1)$  with  $\rho_{H_1}(A) < 0$  or  $H_1$  has a 3-regular subgraph  $H'_1$ . If the latter holds, then  $H'_1$  must contain the edge  $wv$  since  $G$  has no 3-regular subgraphs. Then the neighbors of  $w$  in  $H'_1 - v$  are not pale in  $H_1$  and hence in  $G$ , a contradiction to Claim 5.4 (v). We conclude that there exists a set  $A_1 \subseteq V(H_1)$  with  $\rho_{H_1}(A_1) < 0$ . Since  $\rho_G(A_1) \geq 0$ , we know that  $w, v \in A_1$  and  $u' \notin A_1$ . Then

$$\rho_G(A_1 \cup \{u\}) = \rho_G(A_1) + 3 - 2 \cdot 2 = \rho_{H_1}(A_1) + 2 + 3 - 4 = \rho_{H_1}(A_1) + 1 \geq 0.$$

It follows that  $\rho_{H_1}(A_1) = -1$ ,  $\rho_G(A_1 \cup \{u\}) = \rho_{H_1}(A_1) + 1 = 0$  and  $\rho_G(A_1) = \rho_G(A_1 \cup \{u\}) - 3 + 2(2) = 1$ . Also  $w' \notin A_1$ , since otherwise we have  $\rho_G(A_1 \cup \{u, u'\}) = \rho_G(A_1) + 2 \cdot 3 - 4 \cdot 2 = -1$ , a contradiction.

By symmetric argument, we can also find a set  $A'_1$  such that  $w', v \in A'_1$ ,  $u, w \notin A'_1$ ,  $\rho_G(A'_1 \cup \{u'\}) = 0$  and  $\rho_G(A'_1) = 1$ . Then by Lemma 5.2,

$$\rho_G((A_1 \cup \{u\}) \cap (A'_1 \cup \{u'\})) + \rho_G((A_1 \cup \{u\}) \cup (A'_1 \cup \{u'\})) \leq \rho_G(A_1 \cup \{u\}) + \rho_G(A'_1 \cup \{u'\}) = 0 + 0 = 0.$$

Since  $(A_1 \cup \{u\}) \cap (A'_1 \cup \{u'\}) = A_1 \cap A'_1$ , we conclude that

$$\rho_G(A_1 \cap A'_1) = 0. \tag{5.2}$$

If  $ww' \in E(G)$ , then by Lemma 5.2,

$$\begin{aligned} \rho_G(A_1 \cup A'_1 \cup \{u, u'\}) &= \rho_G(A_1 \cup \{u\}) + \rho_G(A'_1 \cup \{u'\}) - \rho_G((A_1 \cup \{u\}) \cap (A'_1 \cup \{u'\})) \\ &\quad - 2|E_G(\{u\} \cup A_1 \setminus A'_1, \{u'\} \cup A'_1 \setminus A_1)| \\ &\leq 0 + 0 - 0 - 2(1) = -2, \end{aligned}$$

a contradiction. Thus  $ww' \notin E(G)$ .

Now we consider the graph  $H_2$  obtained from  $G - \{wu, w'u'\}$  by adding the edge  $ww'$ . Recall that  $e'$  is the unique edge incident to  $w'$  such that  $e' \notin S(G) \cup S'(G)$ , if it exists. Let  $e''$  be the unique edge incident to  $w$  such that  $e'' \notin S(G) \cup S'(G)$ , if it exists.

Assume  $H_2$  has a  $3\Delta$ -coloring  $f_2$ . Then  $f_2(uv) \neq f_2(u'v)$ . Since  $e'$  and  $e''$  are distance one from each other in  $H_2$ ,  $f_2(e') \neq f_2(e'')$ . We may assume that  $f_2(e') \neq f_2(u'v)$  and  $f_2(e'') \neq f_2(uv)$  by switching the colors of  $uv$  and  $u'v$  if necessary.

Then we let

$$f(e) = f_2(e) \text{ for } e \in E(G) \setminus (S(G) \cup S'(G) \cup B(G)).$$

The only edge in  $N_G^2(uv)$  incident to  $w$  and colored in  $f_2$  is  $e''$ , and the only edge in  $N_G^2(u'v)$  incident to  $w'$  colored in  $f_2$  is  $e'$ . Since  $f_2(e'') \neq f_2(uv)$  and  $f_2(e') \neq f_2(u'v)$ ,  $f$  is a partial  $3\Delta$ -coloring of  $G$ . Now Claim 5.4 (ii) implies that  $(S'(G), S(G), B(G))$  is a  $(f, 3\Delta)$ -degenerate sequence for  $G$ . Thus  $G$  is  $(f, 3\Delta)$ -degenerate, a contradiction. So  $H_2$  must not have a  $3\Delta$ -coloring.

Since  $H_2$  contains fewer  $2^+$ -vertices than  $G$ , this means that  $H_2$  does not satisfy the conditions of our theorem. So either there exists a set  $A_2 \subseteq V(H_2)$  with  $\rho_{H_2}(A_2) < 0$  or  $H_2$  has a 3-regular subgraph  $H'_2$ . In the latter case,  $H'_2$  must contain the edge  $ww'$ , since  $G$  contains no 3-regular subgraphs. Then the neighbors of  $w$  in  $H'_2 - w'$  are not pale in  $H_2$  and hence in  $G$ , a contradiction to Claim 5.4 (v). We conclude that there exists a set  $A_2 \subseteq V(H_2)$  with  $\rho_{H_2}(A_2) < 0$ . Since  $\rho_G(A_2) \geq 0$ , we know that  $w, w' \in A_2$ . We may also assume that  $u, u' \notin A_2$  since  $\rho_{H_2}(A_2 \setminus \{u, u'\}) \leq \rho_{H_2}(A_2)$ . Then  $\rho_G(A_2) = \rho_{H_2}(A_2) + 2 \leq 1$ . Also  $v \notin A_2$  since otherwise  $\rho_G(A_2 \cup \{u, u'\}) = \rho_G(A_2) + 2 \cdot 3 - 4 \cdot 2 \leq -1$ , a contradiction. Thus we have a set  $A_2$  with  $\rho_{H_2}(A_2) \leq 1$ ,  $w, w' \in A_2$  and  $u, v, u' \notin A_2$ .

Since  $\{v, w, w'\} \subseteq A_1 \cup A_2$  and  $\{u, u'\} \cap (A_1 \cup A_2) = \emptyset$ ,  $\rho_G(A_1 \cup A_2 \cup \{u, u'\}) = \rho_G(A_1 \cup A_2) + 2(3) - 4(2) = \rho_G(A_1 \cup A_2) - 2$ . It follows that  $\rho_G(A_1 \cup A_2) \geq 2$ . So by Lemma 5.2,

$$\rho_G(A_1 \cap A_2) = \rho_G(A_1) + \rho_G(A_2) - \rho_G(A_1 \cup A_2) \leq 1 + 1 - 2 = 0.$$

Then by (5.2) and again by Lemma 5.2

$$\rho_G((A_1 \cap A'_1) \cup (A_1 \cap A_2)) \leq \rho_G(A_1 \cap A'_1) + \rho_G(A_1 \cap A_2) = 0 + 0 = 0.$$

Yet,  $(A_1 \cap A'_1) \cup (A_1 \cap A_2)$  contains  $v$  and  $w$  and does not contain  $u$ . So

$$\rho_G((A_1 \cap A'_1) \cup (A_1 \cap A_2) \cup \{u\}) = \rho_G((A_1 \cap A'_1) \cup (A_1 \cap A_2)) + 3 - 2(2) = -1,$$

a contradiction to the choice of  $G$ . This proves the lemma.  $\square$

**Claim 5.6.** *Let  $v \in V(G^*)$ .*

- (i) *If  $d_{G^*}(v) < \Delta$ , then at least one vertex in  $N_{G^*}(v)$  is neither poor nor very poor.*
- (ii) *If  $d_{G^*}(v) = \Delta$ , then either at least one vertex in  $N_{G^*}(v)$  is neither poor nor very poor or  $v$  is not a sponsor of a very poor vertex.*

*Proof.* Let  $d_{G^*}(v) = s$  and  $N_{G^*}(v) = \{u_1, \dots, u_s\}$  where each  $u_i$  is either poor or very poor. Further assume that  $u_1, \dots, u_t$  ( $t \leq s$ ) are the poor  $3G^*$ -vertices in  $N_{G^*}(v)$ . By Claim 5.3,

$$d_{G^*}(x) = d_G(x) \text{ for each } x \in N_{G^*}[v]. \quad (5.3)$$

In particular,  $d_{G^*}(v) = d_G(v) = s$ . For  $i \in [t]$ , since  $u_i$  is a poor  $3G^*$ -vertex, let  $N_G(u_i) = \{v, u'_i, u''_i\}$ , where  $u''_i$  is the unique  $2G^*$ -vertex in  $N_G(u_i)$ ,  $N_G(u''_i) = \{u_i, x_i\}$ , and let  $e_i := u_i u''_i$ . Note that  $e_i$  may or may not be a sink. For  $i \in [s] \setminus [t]$ , we know  $d(u_i) = 2$ , so let  $N_G(u_i) = \{v, u'_i\}$ .

(i) Suppose  $s < \Delta$ . Recall that  $d_G(v) = s < \Delta$ . For each  $i \in [t]$ , the poor  $3G^*$ -vertex  $u_i$  is adjacent to the  $(\Delta - 1)_G^-$ -vertex  $v$  and hence is incident to exactly one sink  $e_i$ .

Let  $H := G - v$ . By the minimality of  $G$ , the graph  $H$  has a  $3\Delta$ -coloring  $f'$ . Let

$$f(e) := f'(e) \text{ for } e \in E(G) \setminus B(G).$$

Then  $f$  is a partial  $3\Delta$ -coloring of  $G$ . Consider the ordering  $(vu_1, \dots, vu_s, B(H))$  of the edges not colored by  $f$ . First, since  $d_G(v) \leq \Delta - 1$ , for each  $i \in [t]$ ,

$$\begin{aligned} |N_G[vu_i] \setminus B(G)| &\leq d_G(u'_i) + d_G(u''_i) + \sum_{j \neq i} |N_G(u_j) \setminus B(G)| + |\{vu_i\}| \\ &\leq \Delta + 2 + 2(d_G(v) - 1) + 1 \leq 3\Delta - 1. \end{aligned}$$

Similarly, if  $t < i \leq s$ , then

$$\begin{aligned} |N_G[vu_i] \setminus (B(G) \cup \{vu_1, \dots, vu_s\})| &\leq d_G(u'_i) + \sum_{j \neq i} |N_G(u_j) \setminus B(G)| + |\{vu_i\}| \\ &\leq \Delta + 2(d_G(v) - 1) + 1 \leq 3\Delta - 3. \end{aligned}$$

Claim 5.4 (ii) implies that  $|N_G[e]| \leq 3\Delta$  for each  $e \in B(G)$ . Therefore,  $(vu_1, \dots, vu_s, B(G))$  is a  $(f, 3\Delta)$ -degenerate sequence for  $G$ . Hence  $G$  is  $(f, 3\Delta)$ -degenerate, a contradiction to Lemma 2.1 and the assumption that  $G$  does not have a  $3\Delta$ -coloring.

(ii) Assume  $s = \Delta$  and  $u_\Delta$  is very poor, so that  $t < s$ . Let  $u'$  be the rival of  $u_\Delta$  and let  $e'$  be the unique edge incident to  $u'$  where  $e' \notin S(G) \cup S'(G)$ , if it exists.

Consider  $H := G - v$ . By the minimality of  $G$ , graph  $H$  has a  $3\Delta$ -coloring  $f'$ . Let

$$f(e) := f'(e) \text{ for } e \in E(G) \setminus (\{e_1, \dots, e_t\} \cup S(G) \cup S'(G) \cup B(G)).$$

Then  $f$  is a partial  $3\Delta$ -coloring of  $G$  since  $N_H^2[e] = N_G^2[e] \setminus \Gamma_G(v)$  for each  $e \in E(H)$ . Let  $B'(G) := B(G) \setminus \{e_1, \dots, e_t\}$  and  $P(G) := S'(G) \cup S(G) \cup B'(G)$ .

Consider the sequence  $(vu_1, \dots, vu_{\Delta-1}, e_1, \dots, e_t, vu_\Delta, S'(G), S(G), B'(G))$  of edges not colored by  $f$ . We want to show that it is a  $(f, 3\Delta)$ -degenerate sequence for  $G$ . Note that no edge incident to  $u_\Delta$  is colored by  $f$ . First, for  $i \in [t]$ ,

$$\begin{aligned} |N_G^2[vu_i] \setminus (\{e_1, \dots, e_t, vu_\Delta\} \cup P(G))| &\leq d_G(u'_i) + (d_G(u''_i) - 1) + \sum_{j \notin \{i, \Delta\}} |N_G(u_j) \setminus \{e_j\}| + |\{vu_i\}| \\ &\leq \Delta + 1 + 2(\Delta - 2) + 1 \leq 3\Delta - 2. \end{aligned}$$

If  $t < i \leq \Delta - 1$ , then

$$\begin{aligned} |N_G^2[vu_i] \setminus (\{e_1, \dots, e_t, vu_\Delta\} \cup P(G))| &\leq d_G(u'_i) + \sum_{j \notin \{i, \Delta\}} |N_G(u_j) \setminus \{e_1, \dots, e_t\}| + |\{vu_i\}| \\ &\leq \Delta + 2(\Delta - 2) + 1 \leq 3\Delta - 3. \end{aligned}$$

For  $i \in [t]$ ,

$$\begin{aligned} |N_G^2[e_i] \setminus (\{vu_\Delta\} \cup P(G))| &\leq d_G(x_i) + d_G(u'_i) + |N_G(v) \setminus \{vu_\Delta\}| + |\{e_i\}| \\ &\leq \Delta + \Delta + \Delta - 1 + 1 \leq 3\Delta. \end{aligned}$$

Also

$$|N_G^2[vu_\Delta] \setminus P(G)| \leq \sum_{i=1}^{\Delta-1} d_G(u_i) + |\{e'\}| + |\{vu_\Delta\}| \leq 3(\Delta - 1) + 2 \leq 3\Delta - 1.$$

These inequalities together with Claim 5.4 (iii) imply that the sequence

$(vu_1, \dots, vu_{\Delta-1}, e_1, \dots, e_t, vu, S'(G), S(G), B'(G))$  is  $(f, 3\Delta)$ -degenerate sequence for  $G$ . So,  $G$  is  $(f, 3\Delta)$ -degenerate, a contradiction to Lemma 2.1 and the choice of  $G$ . This proves the claim.  $\square$

**Claim 5.7.** *Suppose  $v$  is a poor  $3G^*$ -vertex and  $N_{G^*}(v) = \{v_1, v_2, v_3\}$ , where  $d_{G^*}(v_1) = 4$ ,  $d_{G^*}(v_2) = 3$ , and  $d_{G^*}(v_3) = 2$ . Then  $v_1$  has at least two neighbors in  $G^*$  where each of them is neither poor nor very poor.*

*Proof.* Suppose that under the conditions of the lemma,  $N_{G^*}(v_1) = \{v, u_1, u_2, x\}$ , where each of  $u_1$  and  $u_2$  is either poor or very poor. Let  $N_{G^*}(v_2) = \{v, y_1, y_2\}$  and  $N_{G^*}(v_3) = \{v, z\}$ . Since  $v, u_1$ , and  $u_2$  are all poor or very poor, their degrees in  $G$  are the same as in  $G^*$ . Since  $\Delta \geq 7$ ,

$$2\Delta + d_{G^*}(v_1) = 2\Delta + 4 > 10 + \Delta > 3 + 3 + 3 + \Delta \geq d_G(v) + d_G(u_1) + d_G(u_2) + d_G(x).$$

Thus Claim 5.3 (i) implies that  $d_G(v_1) = d_{G^*}(v_1) = 4$ . Also by Claim 5.3 (iv),  $d_G(v_2) = d_{G^*}(v_2) = 3$ .



Consider  $H := G - v$ . By the minimality of  $G$ , the graph  $H$  has a  $3\Delta$ -coloring  $f$ . By the construction of  $H$ ,  $f$  is a partial  $3\Delta$ -coloring of  $G$ .

Since  $\Delta \geq 7$ ,

$$\begin{aligned} |N_G^2[vv_1] \setminus \{vv_2, vv_3\}| &\leq d_G(x) + (d_G(v_2) - 1) + (d_G(v_3) - 1) + \sum_{i=1}^2 d_G(u_i) + |\{vv_1\}| \\ &\leq \Delta + 2 + 1 + 2 \cdot 3 + 1 \leq 3\Delta, \\ |N_G^2[vv_2] \setminus \{vv_3\}| &\leq d_G(y_1) + d_G(y_2) + d_G(v_1) + d_G(v_3) - 1 + |\{vv_2\}| \\ &\leq 2\Delta + 4 + 1 + 1 \leq 3\Delta, \\ |N_G^2[vv_3]| &\leq d_G(z) + d_G(v_2) + d_G(v_1) + |\{vv_3\}| \leq \Delta + 3 + 4 \leq 3\Delta. \end{aligned}$$

Thus  $(vv_1, vv_2, vv_3)$  is a  $(f, 3\Delta)$ -degenerate sequence for  $G$ . So  $G$  is  $(f, 3\Delta)$ -degenerate, a contradiction to the choice of  $G$ .  $\square$

**5.3. Discharging.** Since  $G^*$  is a subgraph of  $G$ , we know  $\text{Mad}(G^*) \leq 3$  and  $G^*$  does not contain 3-regular subgraphs. For every  $v \in V(G^*)$ , define the *initial charge*  $\text{ch}(v) := d_{G^*}(v)$ . Since  $\text{Mad}(G^*) \leq 3$ , we have  $\sum_{v \in V(G^*)} \text{ch}(v) \leq 3|V(G^*)|$ . We will move the charge among vertices without changing the total sum of charge according to the discharging rules below.

- (R1) If a  $\Delta_{G^*}$ -vertex  $v$  is a sponsor of a very poor vertex  $u$ , then  $v$  gives charge 1 to  $u$ .
- (R2) If  $5 \leq d_{G^*}(v) \leq \Delta$  and  $u$  is a poor vertex in  $N_{G^*}(v)$ , then  $v$  gives charge  $1/2$  to  $u$ .
- (R3) If a  $4_{G^*}$ -vertex  $v$  is adjacent to a poor  $2_{G^*}$ -vertex  $u$ , then  $v$  gives charge  $1/2$  to  $u$ .
- (R4) If a  $4_{G^*}$ -vertex  $v$  is adjacent to a poor  $3_{G^*}$ -vertex  $u$ , then we do one of the following:
  - (R4A) If  $N_{G^*}(v)$  contain at least two vertices that are neither poor nor very poor, then  $v$  gives charge  $1/2$  to  $u$ .
  - (R4B) Otherwise,  $v$  gives charge  $1/4$  to  $u$ .
- (R5) If a poor  $3_{G^*}$ -vertex  $v$  is adjacent to a poor  $2_{G^*}$ -vertex  $u$ , then  $v$  gives charge  $1/2$  to  $u$ .

For every  $v \in V(G^*)$ , let  $\text{ch}^*(v)$  be the *final charge* of  $v$ , which is the charge of  $v$  after the distribution. Since the total sum of charge did not change,

$$\sum_{v \in V(G^*)} \text{ch}^*(v) = \sum_{v \in V(G^*)} \text{ch}(v) \leq 3|V(G^*)|. \quad (5.4)$$

See Figure 2 for an illustration of the discharging rules.

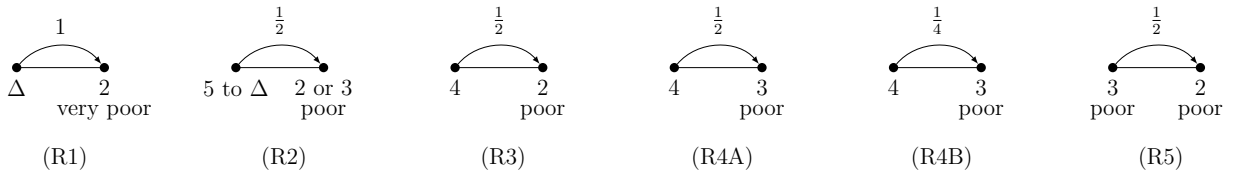


FIGURE 2. The discharging rules.

The next claim shows important properties of the final charge  $\text{ch}^*$ .

**Claim 5.8.** *Let  $v \in V(G^*)$ .*

- (i) *If  $v$  is a  $2_{G^*}$ -vertex, then  $\text{ch}^*(v) = 3$ .*
- (ii) *If  $v$  is a  $3_{G^*}$ -vertex, then  $\text{ch}^*(v) \geq 3$ .*
- (iii) *If  $v$  is a  $4_{G^*}$ -vertex, then  $\text{ch}^*(v) \geq 3$ . Moreover, if  $\text{ch}^*(v) = 3$ , then  $v$  is adjacent to exactly two poor vertices and no very poor vertices.*

- (iv) If  $5 \leq d_{G^*}(v) \leq \Delta - 1$ , then  $\text{ch}^*(v) \geq 3$ . Moreover, if  $\text{ch}^*(v) = 3$ , then  $d_{G^*}(v) = 5$  and  $v$  has exactly four poor neighbors.
- (v) If  $v$  is a  $\Delta_{G^*}$ -vertex, then  $\text{ch}^*(v) > 3$ .

*Proof.* (i) Suppose  $d_{G^*}(v) = 2$ . Then  $v$  is either poor or very poor. If  $v$  is very poor, then it is adjacent to its sponsor  $w$ , and by Claim 5.4 (iii),  $d_{G^*}(w) = \Delta$ . Hence by (R1),  $w$  gives charge 1 to  $v$ , so  $\text{ch}^*(v) = 3$ .

If  $v$  is poor but not very poor, then both neighbors of  $v$  are  $3_{G^*}^+$ -vertices, and each  $3_{G^*}$ -neighbor of  $v$  is poor. Thus, by rules (R2), (R3), or (R5),  $v$  receives  $1/2$  from each of its two neighbors in  $G^*$ . So  $\text{ch}^*(v) = 3$ .

(ii) Suppose  $d_{G^*}(v) = 3$ . If  $v$  is not poor, then  $v$  does not send or receive any charge, so  $\text{ch}^*(v) = \text{ch}(v) = 3$ . Now assume  $v$  is poor. Then by Claim 5.4 (iv),  $v$  has a  $4_{G^*}^+$ -neighbor. By (R5),  $v$  gives  $1/2$  to its unique  $2_{G^*}$ -neighbor, and by (R2) or (R4) receives at least  $1/4$  from each  $4_{G^*}^+$ -neighbor. Thus, if  $\text{ch}^*(v) < 3$ , then  $v$  has only one  $4_{G^*}^+$ -neighbor, say  $u$ . Furthermore, by (R2),  $d_{G^*}(u) = 4$ , and by (R4),  $u$  has at most one neighbor in  $G^*$  that is neither poor nor very poor. But this contradicts Claim 5.7.

(iii) Suppose  $d_{G^*}(v) = 4$ . If  $v$  is the rival of a very poor vertex  $u$ , then by the definition (T3), every  $2_{G^*}$ -neighbor of  $v$  is very poor and hence  $v$  does not give to its  $2_{G^*}$ -neighbors anything. In this case, it either gives  $1/2$  to its unique  $3_{G^*}$ -neighbor by (R4A) or gives  $1/4$  to each of its at most three poor neighbors by (R4B). In both cases,  $\text{ch}^*(v) \geq 4 - 3 \cdot \frac{1}{4} > 3$ .

Otherwise,  $v$  is not adjacent to any very poor vertices. Then by Claim 5.6 (i),  $v$  is adjacent to at most three poor vertices. If  $v$  has three poor neighbors, then all of them are  $3_{G^*}$ -vertices, because in this case each  $2_{G^*}$ -neighbor of  $v$  is very poor. Thus  $v$  sends charge  $1/4$  to its three poor neighbors by (R4B), so  $\text{ch}^*(v) \geq 4 - 3 \cdot \frac{1}{4} > 3$ . The last possibility is that  $v$  has at most two poor neighbors. Since  $v$  gives to any poor neighbor at most  $1/2$ , the only possibility to have  $\text{ch}^*(v) \leq 3$  is that  $v$  has exactly two poor neighbors and gives  $1/2$  to each of them, in which case  $\text{ch}^*(v) = 3$ . This proves (iii).

(iv) Suppose  $5 \leq d_{G^*}(v) \leq \Delta - 1$ . By (R2),  $v$  gives charge  $1/2$  to each of its poor neighbors. By Claim 5.6 (i), the number of such neighbors is at most  $d_{G^*}(v) - 1$ . So

$$\text{ch}^*(v) \geq \text{ch}(v) - \frac{1}{2}(d_{G^*}(v) - 1) = \frac{d_{G^*}(v) + 1}{2} \geq 3.$$

Moreover, if  $\text{ch}^*(v) = 3$ , then  $d_{G^*}(v) = 5$  and  $v$  is adjacent to exactly four poor neighbors.

(v) Suppose  $d_{G^*}(v) = \Delta$ . If  $v$  has no very poor neighbors, then by (R2) it gives at most  $1/2$  to each of its neighbors, and  $\text{ch}^*(v) \geq \text{ch}(v) - \frac{d_{G^*}(v)}{2} = \frac{d_{G^*}(v)}{2} \geq \frac{\Delta}{2} > 3$ . Otherwise, by Claim 5.5, it has only one very poor neighbor (to which it gives 1 by (R1)), but then by Claim 5.6 (ii), it has a neighbor that is neither poor nor very poor. Thus by (R1) and (R2), again

$$\text{ch}^*(v) \geq d_{G^*}(v) - 1 - \frac{\Delta - 2}{2} = \Delta/2 > 3.$$

□

Claim 5.8 implies that  $\sum_{v \in V(G^*)} \text{ch}^*(v) \geq 3|V(G^*)|$  and together with (5.4) yields

$$\text{ch}^*(v) = 3 \text{ for each } v \in V(G^*). \quad (5.5)$$

This yields the following facts.

**Claim 5.9.** *Graph  $G$  has the following properties.*

- (i)  $\Delta(G^*) \leq 5$ .
- (ii)  $G^*$  has no very poor vertices.

- (iii)  $B(G) = \emptyset$ .
- (iv) If  $v$  is a  $5_{G^*}$ -vertex, then  $d_G(v) = d_{G^*}(v)$ .
- (v) There are no  $5_{G^*}$ -vertices; in other words,  $\Delta(G^*) \leq 4$ .
- (vi) There are no poor  $3_{G^*}$ -vertices.

*Proof.* (i) Claim (i) follows from (5.5) and Claim 5.8 (parts (iv) and (v)).

(ii) If  $v$  is a very poor vertex in  $G^*$ , then by Claim 5.4 (iii), it has a  $\Delta_{G^*}$ -neighbor, contradicting (i).

(iii) Suppose that  $G^*$  has a  $u$ -sink  $uw$ , which means that  $d_{G^*}(w) = 2$ ,  $d_{G^*}(u) = 3$ , and

$$d_G(u') + d_G(u'') \leq 2\Delta - 1. \quad (5.6)$$

where  $N_{G^*}(u) = \{w, u', u''\}$  and  $N_{G^*}(w) = \{u, w'\}$ . Recall that by Claim 5.3,  $d_G(w) = d_{G^*}(w) = 2$  and  $d_G(u) = d_{G^*}(u) = 3$ . Let  $H$  be obtained from  $G - uw$  by adding leaves adjacent only to  $w'$  so that the degree of  $w'$  in  $H$  is  $\Delta$ . Since  $d_H(w) = 1$ ,  $H$  has fewer  $2^+$ -vertices than  $G$ . Also,  $\text{Mad}(H) \leq 3$  and  $H$  has no 3-regular subgraphs. So by the minimality of  $G$ , the graph  $H$  has a  $3\Delta$ -coloring  $f$ . By (i) and the fact that  $d_H(w) = 1$ , vertex  $w'$  has at least  $\Delta - 4 \geq 3$   $1_H$ -neighbors, including  $w$ . Hence we can switch the colors of the pendant edges incident to  $w'$  so that  $f(w'w) \notin \{f(uw'), f(uw'')\}$ . Then  $f|_{E(G)}$  is a partial  $3\Delta$ -coloring of  $G$ , where the only non-colored edge is  $uw$ . But by (5.6),

$$|N_G^2[uw]| \leq d_G(u') + d_G(u'') + d_G(w') + 1 \leq (2\Delta - 1) + \Delta + 1 = 3\Delta,$$

so we can extend  $f$  to  $uw$ , a contradiction to the choice of  $G$ . This proves (iii).

(iv) Suppose  $d_{G^*}(v) = 5$ . By Claim 5.8 (iv), we may assume that  $N_{G^*}(v) = \{u_1, \dots, u_5\}$ , where each of  $u_1, u_2, u_3, u_4$  is either poor or very poor. In particular, by Claim 5.3,  $d_G(u_i) = d_{G^*}(u_i) \leq 3$  for  $i \in [4]$ . Thus,

$$\sum_{i=1}^5 d_G(u_i) \leq 3(4) + \Delta \leq 5 + 2\Delta = d_{G^*}(v) + 2\Delta,$$

and Claim 5.3 (i) implies  $d_G(v) = d_{G^*}(v)$ .

(v) Suppose  $d_{G^*}(v) = 5$  and  $N_{G^*}(v) = \{u_1, \dots, u_5\}$ , where each of  $u_1, u_2, u_3, u_4$  is either poor or very poor. If some  $u_i$  is a poor  $3_{G^*}$ -vertex adjacent to a  $2_{G^*}$ -vertex  $w_i$ , then by (iv),  $u_i w_i$  is the  $u_i$ -sink. But this contradicts (iii). Thus

$$\text{for each } i \in [4], u_i \text{ is a poor } 2_{G^*}\text{-vertex.} \quad (5.7)$$

So we may assume that  $N_G(u_i) = \{v, w_i\}$  for  $i \in [4]$ . Let  $H$  be obtained from  $G - vu_1$  by adding leaves adjacent only to  $w_1$  so that the degree of  $w_1$  in  $H$  is  $\Delta$ . Since  $d_H(u_1) = 1$ ,  $H$  has fewer  $2^+$ -vertices than  $G$ . Also,  $\text{Mad}(H) \leq 3$  and  $H$  has no 3-regular subgraphs. So by the minimality of  $G$ , the graph  $H$  has a  $3\Delta$ -coloring  $f$ . By (i) and the fact that  $d_H(u_1) = 1$ , the number of  $1_H$ -neighbors of the vertex  $w_1$ , including  $u_1$ , is at least  $\Delta - 4 \geq 3$ . Hence we can switch the colors of the pendant edges incident to  $w_1$  so that  $f(w_1 u_1) \neq f(v u_5)$ .

If  $f(w_1 u_1) \notin \{f(v u_2), f(v u_3), f(v u_4)\}$ , then  $f|_{E(G)}$  is a partial  $3\Delta$ -coloring of  $G$ , where the only uncolored edge is  $u_1 v$ . But in this case by (5.7),

$$|N_G^2[u_1 v]| \leq d_G(w_1) + \sum_{i=2}^5 d_G(u_i) + 1 \leq \Delta + 3(2) + \Delta + 1 = 2\Delta + 7 \leq 3\Delta, \quad (5.8)$$

so we can extend  $f$  to  $u_1 v$ , a contradiction to the choice of  $G$ .

Thus by symmetry, we may assume  $f(w_1 u_1) = f(v u_2)$ . Then the coloring  $f'$  obtained from  $f|_{E(G)}$  by uncoloring  $vu_2$  is a partial  $3\Delta$ -coloring of  $G$ , where the only uncolored edges are  $u_1 v$  and  $u_2 v$ .

Again by (5.8), we can extend  $f'$  to  $u_1v$  and then by the similar inequality for  $|N_G^2[u_2v]|$ , extend it to  $u_2v$ , a contradiction to the choice of  $G$ . This proves (v).

(vi) Suppose that  $v$  is a poor  $3_{G^*}$ -vertex with  $N_{G^*}(v) = \{u_1, u_2, u_3\}$  where  $d_{G^*}(u_1) = 2$  and  $d_{G^*}(u_2), d_{G^*}(u_3) \geq 3$ . By (v),  $d_{G^*}(u_2), d_{G^*}(u_3) \leq 4$ . Moreover, if, say,  $d_{G^*}(u_2) = 3$ , then by Claim 5.3 (iv),  $d_G(u_2) = 3$ . Hence  $vu_1$  is a  $v$ -sink, yet, this contradicts (iii). Therefore  $d_{G^*}(u_2) = d_{G^*}(u_3) = 4$ . So, by Claim 5.8 (iii), each of  $u_2$  and  $u_3$  has exactly two poor neighbors in  $G^*$ . Now by (R4), each of  $u_2$  and  $u_3$  gives  $1/2$  to  $v$ , while  $v$  gives to  $u_1$  only  $1/2$  by (R5). Hence  $\text{ch}^*(v) = 3 + 2 \cdot \frac{1}{2} - 1/2 = 7/2 > 3$ , a contradiction to (5.5).  $\square$

Claim 5.9 implies that the degree of a vertex in  $G^*$  must be in  $\{2, 3, 4\}$ . Moreover, since  $G^*$  has neither very poor vertices nor poor  $3_{G^*}$ -vertices and each  $4_{G^*}$ -vertex is adjacent to exactly two poor vertices,

$$\text{each } 2_{G^*}\text{-vertex is adjacent only to } 4_{G^*}\text{-vertices, and each } 4_{G^*}\text{-vertex is adjacent to exactly two } 2_{G^*}\text{-vertices.} \quad (5.9)$$

The last claim that we need is:

**Claim 5.10.** *If  $v$  is a  $4_{G^*}$ -vertex, then  $d_G(v) = d_{G^*}(v)$ .*

*Proof.* Let  $v$  be a  $4_{G^*}$ -vertex. By (5.9), we may assume that  $N_{G^*}(v) = \{u_1, u_2, u_3, u_4\}$ , where  $d_G(u_1) = d_G(u_2) = 2$ . So

$$\sum_{i=1}^4 d_G(u_i) \leq 2(2) + 2\Delta = 2\Delta + d_{G^*}(v).$$

Hence by Claim 5.3 (i),  $d_G(v) = d_{G^*}(v) = 4$ .  $\square$

Now we are ready to finish the proof of Theorem 5.1. Since  $\text{Mad}(G^*) \leq 3$  and  $G^*$  has no 3-regular subgraphs, it has a vertex  $u$  with  $d_G(u) = d_{G^*}(u) = 2$ . Let  $N_G(u) = \{v, w\}$ . By (5.9),  $d_{G^*}(v) = d_{G^*}(w) = 4$ . So by Claim 5.10,  $d_G(v) = d_G(w) = 4$ . Let  $N_G(v) = \{u, v_1, v_2, v_3\}$ . By (5.9), we may assume that  $d_G(v_1) = 2$ .

Let  $H$  be the graph obtained from  $G - uv$  by adding  $\Delta - 4$  leaves adjacent only to  $w$  so that  $d_H(w) = \Delta$ . Since  $d_H(u) = 1$ ,  $H$  has fewer  $2^+$ -vertices than  $G$ . Also,  $\text{Mad}(H) \leq 3$  and  $H$  has no 3-regular subgraphs. So by the minimality of  $G$ , the graph  $H$  has a  $3\Delta$ -coloring  $f$ . Since  $d_G(w) = 4$ , the number of  $1_H$ -neighbors of the vertex  $w$ , including  $u$ , is at least  $\Delta - 3 \geq 4$ . Hence we can switch the colors of the pendant edges incident to  $w$  so that  $f(wu) \notin \{f(vv_1), f(vv_2), f(vv_3)\}$ . Thus  $f|_{E(G)}$  is a partial  $3\Delta$ -coloring of  $G$ , where the only uncolored edge is  $uv$ . However,

$$|N_G^2[uv]| \leq d_G(w) + \sum_{i=1}^3 d_G(v_i) + 1 \leq 4 + 2 + 2\Delta + 1 = 2\Delta + 7 \leq 3\Delta,$$

and so we can extend  $f$  to  $uv$ , a contradiction to the choice of  $G$ . This finishes the proof of Theorem 5.1.

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